The use of Mean Residual Life in testing departures from Exponentiality

S.Rao Jammalamadaka
Department of Statistics and Applied Probability
University of California
Santa Barbara, CA 93106-3110

Emanuele Taufer
Department of Computer and Management Sciences
University of Trento
Trento, I 38100.

Summary. We make use of the characterization that $E(X - t|X > t)$ is constant over $t \in [0, \infty)$ if and only if X is distributed as an exponential r.v., in order to build new test statistics for exponentiality. We analyze the asymptotic properties of the proposed procedures. Simulation studies indicate that the proposed statistics have very good power in a large variety of situations.

Keywords Mean Residual Life, Test for exponentiality, Kolmogorov-Smirnov statistic, Cramer-von Mises statistic, Wiener process, Quantile process.

1 Introduction

Mean residual life (MRL) is a very well-known and central concept in reliability and survival analysis; if X denotes a non negative random variable (r.v.) with distribution function (d.f.) F, then the MRL at time t is defined as

$$m(t) = E(X - t|X > t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} \tag{1.1}$$

where $\bar{F} = 1 - F$ is the survival function. If $m(t)$ is non-increasing (or non-decreasing) in t, then X is said to have a decreasing (or increasing) MRL distribution (DMRL or IMRL respectively). MRL is related to the hazard rate $\lambda(t)$ by

$$\lambda(t) = \frac{1 + m'(t)}{m(t)} \tag{1.2}$$

and it can be shown that the class of DMRL (IMRL) distributions include the class of increasing (decreasing) failure rate distributions (IFR and DFR respectively). For a review of these concepts and other properties of MRL, we refer the reader to Patel (1985) and Guess and Proschan (1988).

The exponential distribution is a natural boundary between DMRL and IMRL distributions, having a constant mean residual life. Indeed, it has been shown by Shanbhag (1970) that the exponential distribution can be characterized by constancy of MRL, more precisely, if F is exponential with mean θ it holds that

$$m(t) = \theta, \quad \forall t. \tag{1.3}$$

The above characterization can be easily seen to be equivalent to either

$$\int_t^\infty \bar{F}(x)dx = \bar{F}(t)\,\theta, \quad \forall t \tag{1.4}$$

or

$$E[\min(X, t)] = F(t)\,\theta \quad \forall t. \tag{1.5}$$

Characterizations based on MRL have been used to build tests for exponentiality against specialized as well as omnibus alternatives. Hollander and Proschan (1975), Bergman and Klefsjö (1989) and Bandyopadhyay and Basu (1990) provide tests against DMRL distributions. Koul (1978) and Bhattacharjee and Sen (1995) provide tests for the larger class of New Better that Used in Expectation (NBUE) alternatives (recall a distribution is NBUE if the equal sign in (1.4) is replaced by a $<$). More recently Baringhaus and Henze (2000) and

Taufer (2000) provide omnibus tests for exponentiality. All of the above contributions have been proposed by exploiting equations (1.4) and (1.5); one main reason for doing so is to get rid of the denominator $\bar{F}(t)$ in the definition of $m(t)$, so that the test statistics become less complicated.

In the following we plan to utilize directly equation (1.3) in order to provide new tests statistics for exponentiality. Our approach is quite different from that undertaken in the above mentioned works and, as we will see, leads to some powerful alternative tests for exponentiality.

Note that testing for exponentiality still attracts considerable attention and is the topic of a good amount of recent research; beyond the above mentioned contributions we find several other authors who provide new test statistics for detecting departures from the hypothesis of exponentiality along specific or general alternatives. Alwasel (2001), Ahmad and Alwasel (1999) use the lack of memory property of the exponential distribution. Klar (2001) exploits the integrated distribution function while Jammalamadaka and Taufer (2001) consider a characterization based on normalized spacings. Grzegorzewsi and Wieczorkowski (1999) and Ebrahimi et Al. (1992) make use of the maximum entropy principle. Other omnibus tests for exponentiality have been developed by Henze and Meintanis (2002), Henze (1993), Baringhaus and Henze (1991, 1992) who use estimators of the Laplace transform.

Specialized tests are provided by Del Castillo and Puig (1999a, 1999b), Gatto and Jammalamadaka (2002), and Klar (2000) who extends the work of Jammalamadaka and Lee (1998). For a review of earlier contributions, the interested reader is referred to Shapiro (1995), Ascher (1990) and Doksum and Yandell (1984).

2 Test Statistics and their properties

2.1 Construction of a test statistic

Let $X_1, \dots X_{n+1}$ be a random sample from a distribution F with order statistics, $X_{(1)} \leq \dots \leq X_{(n+1)}$ and suppose we wish to test the hypothesis

$$H_0 : F(x) = 1 - e^{-x/\theta}, \theta > 0; \quad vs. \quad H_1 : F(x) \neq 1 - e^{-x/\theta}, \theta > 0.$$

In order to exploit the characterization $m(t) = \theta$, $\forall t$ under exponentiality, define the "sample mean residual life after $X_{(k)}$" as

$$\bar{X}_{>k} = \frac{1}{n-k+1} \sum_{i=k+1}^{n+1} (X_{(i)} - X_{(k)}) \tag{2.6}$$

$$= \frac{1}{n-k+1} \sum_{i=k+1}^{n+1} (n-i+2)(X_{(i)} - X_{(i-1)}). \tag{2.7}$$

For convenience denote the normalized spacings

$$Y_i = (n - i + 2)(X_{(i)} - X_{(i-1)}), \qquad i = 1 \dots n + 1.$$

Observe that under the null hypothesis of exponentiality, we have that $E(\bar{X}_{>k}) = E(\bar{X}) = \theta$, $k = 1, \dots n$. Therefore if we plot the sequence $\bar{X}, \bar{X}_{>1}, \dots \bar{X}_{>n}$ on a chart, under H_0 this should be approximately constant around the true (unknown) value θ. This intuitive graphical approach would suggest to use a distance measure between these "residual sample means" in order to build a test statistic for H_0. One simple and natural way to do this is to exploit a Kolmogorov-Smirnov type distance, that is, reject H_0 when

$$T_n = \max_{1 \le k \le n} \frac{|\bar{X} - \bar{X}_{>k}|}{\bar{X}} \tag{2.8}$$

is large. Note that division by the sample mean is necessary in order to make T_n scale free. We point out however that, as it is, T_n, under the null hypothesis, does not converge to 0; this may be immediately seen if we note that in particular, $\bar{X}_{>n} = Y_{n+1}$ which is exponentially distributed with mean θ under the null hypothesis, no matter what the sample size is. To consider the behavior of T_n a bit more carefully, let $S(i) = \sum_{j \le i} \xi_j$ where ξ_j are *i.i.d.* exponential r.v.'s with mean 1 and $i = n - k + 1$ then

$$\begin{aligned}
T_n &= \max_{1 \le i \le n} \left| 1 - \frac{\bar{X}_{>n-i+1}}{\bar{X}} \right| &(2.9)\\
&\stackrel{D}{=} \max_{1 \le i \le n} \left| 1 - \frac{S(i)}{i} \frac{(n+1)}{S(n+1)} \right| &(2.10)\\
&\le \max_{1 \le i \le n} \left| 1 - \frac{S(i)}{i} \right| + \max_{1 \le i \le n} \frac{S(i)}{i} \left| 1 - \frac{(n+1)}{S(n+1)} \right| &(2.11)\\
&= \max_{1 \le i \le \infty} \left| 1 - \frac{S(i)}{i} \right| + o_p(1) &(2.12)
\end{aligned}$$

as $n \to \infty$. These facts might bring into question the efficiency of T_n in testing for exponentiality.

However, before going further, we try to connect T_n with other test statistics already proposed in the literature. Note that $\bar{X}_{>k}$ is the total time on test transform (TTT) after $X_{(k)}$ divided by $n - k + 1$, i.e. the empirical distribution function (edf) evaluated at t, $t \in [X_{(k)}, X_{(k+1)})$. If we denote the TTT statistic as

$$D_{n+1}(t) = \sum_{i=1}^{k} Y_i + (n - k + 1)(t - X_{(k)}), \quad t \in [X_{(k)}, X_{(k+1)}) \tag{2.13}$$

then, after some manipulation we can rewrite

$$T_n = \max_{1 \le k \le n} \frac{n+1}{n-k+1} \left| \frac{D_{n+1}(X_{(k)})}{(n+1)\bar{X}} - \frac{k}{n+1} \right|. \tag{2.14}$$

One may compare these statistics with those proposed by Koul (1978) and later by Bhattacharjee and Sen (1995), to test against NBUE alternatives in uncensored and censored cases respectively and also with Baringhaus and Henze (2000) for testing H_0 against omnibus alternatives. The key feature of T_n is the weight $(n-k+1)^{-1}$ which comes up naturally in our approach to the problem. The question of interest here is, of course, if this approach can be more fruitful, especially since some power simulations we ran indicated that T_n does not have good power for certain alternatives to exponentiality. The reason, perhaps, is to be searched in the 'high' variance of the last residual means.

This observation and the desire to overcome the problems noted for T_n motivates the construction of a *trimmed* test statistics where some of the last residual means are discarded from T_n. This has to be done in such a way to be able to estimate $m(t)$ over the whole real line. With that in mind, we define

$$T_n^\alpha = \max_{1 \le k < n - n^\alpha + 1} \frac{|\bar{X} - \bar{X}_{>k}|}{\bar{X}}, \quad \alpha \in (0,1).$$

We see that the comparison of the sequence of the residual means goes up to the term with index $n - [n^\alpha]$ where α is a parameter which determines the number of 'later' residual means to be discarded and $[n^\alpha]$ denotes the greatest integer in n^α. It is our intention to investigate properties of the statistic T_n^α. We may also consider statistics based on a quadratic distance between the sample mean and the residual sample means. However, from the discussion above, we would expect to encounter similar problems at the upper tail. We therefore propose to consider a "trimmed" quadratic test statistic, viz.

$$V_n^\alpha = \sum_{k < n - n^\alpha + 1} \left[\frac{\bar{X} - \bar{X}_{>k}}{\bar{X}} \right]^2$$

Remark. A nice bonus of our approach is that the statistics T_n^α and V_n^α can be straightforwardly adapted to the (more general) case

$$H_0 : F = 1 - e^{-(x-\theta_1)/\theta_2}; \quad vs. \quad H_1 : F \ne 1 - e^{-(x-\theta_1)/\theta_2}, \quad \theta_1 \in R, \theta_2 > 0, x \ge \theta_1.$$

by simply replacing $\bar{X}$ with $\bar{X}_{>1}$.

2.2 Asymptotic properties

In terms of the asymptotic properties of the statistics T_n^α and V_n^α, we first note that from (2.7), there are at least $[n^\alpha] + 1$ terms in $\bar{X}_{>k}$ for $1 \le k < n - n^\alpha + 1$. Under H_0, the normalized spacings are exponentially distributed with mean θ hence we have

$$\bar{X}_{>k} \overset{a.s.}{\to} \theta \qquad 1 \leq k < n - n^{\alpha} + 1,$$

by the strong law of large numbers, the speed of convergence depending on the choice of the parameter α. From this it follows that, under exponentiality of the observations

$$T_n^{\alpha} \overset{a.s.}{\to} 0, \quad \text{and} \quad V_n^{\alpha} \overset{a.s.}{\to} 0.$$

The convergence properties of T_n^{α} and V_n^{α} can be studied under more general conditions, which we do in the following theorem.

Theorem 1. *Let $m(t) < \infty$ and F be a continuous d.f. with mean θ, then*

$$\max_{1 \leq k < n - n^{\alpha}+1} |\bar{X} - \bar{X}_{>k}| \overset{p}{\to} \sup_{0 \leq t < \infty} |\theta - m(t)| \tag{2.15}$$

$$\sum_{k < n - n^{\alpha}+1} \left(\bar{X} - \bar{X}_{>k}\right)^2 \overset{p}{\to} \int_0^{\infty} (\theta - m(t))^2 \, dF(t) \tag{2.16}$$

as $n \to \infty$.

The proof of this theorem relies on previous results obtained by Koul (1978) and is postponed to the end of the paper. Under the conditions of Theorem 1 we also have that $\bar{X} \overset{a.s.}{=} \theta$ and hence we immediately obtain the convergence properties of our tests statistics for any class of alternatives with finite mean.

As for the asymptotic distribution of T_n^{α} and V_n^{α}, we have the following result

Theorem 2. *Let $\alpha \in (0, 1)$, then under H_0*

$$n^{\alpha/2} T_n^{\alpha} \overset{D}{\to} \sup_{0 \leq t \leq 1} |W(t)| \tag{2.17}$$

where $W(t)$ is a Wiener process, and

$$2 \left(\log \frac{n+1}{n^{\alpha}} \right)^{-1/2} \left[(n+2) V_n^{\alpha} - \log \frac{n+1}{n^{\alpha}} \right] \overset{D}{\to} N(0,1) \tag{2.18}$$

where $N(0, 1)$ indicates a standard normal r.v..

The result of Theorem 2 shows us that the appropriate normalizing constant depend on the parameter α. The proof of the theorem relies on asymptotic results for functionals of the uniform quantile process in a weighted metric, its proof being postponed to the last section. From the proof of Theorem 2 we will see that result (2.17) still holds if we substitute n^{α} by any $na(n)$ such that $na(n) \to \infty$ and $a(n) \to 0$ as $n \to \infty$. Some simulations we ran indicate that convergence to the standard normal r.v. maybe quite slow in (2.18).

Remark. The null distributions of T_n^α and V_n^α depend on that of an ordered uniform random sample. Hence, as noted by Gupta and Richards (1997) the distributions of T_n^α and V_n^α, sharing the same invariance property with several other tests for exponentiality, remain the same for all random vectors $X_1, \dots X_{n+1}$ having a multivariate Liouville distribution.

3 Monte Carlo Power Estimates

We compare the power performances of our test statistics by the method of Monte Carlo. We generated 10.000 samples of size 10, 30 and 50 for several common alternative distributions which include the Weibull (W) and the Log Normal (LN) with scale parameter 1 and shape parameter θ, linear increasing failure rate distributions (LIFR) with density $(1+\theta x)\exp\{-(x+\theta x^2/2)\}\mathbf{1}_{(x\geq 0)}$, J-shaped (JS) distributions with density $(1+\theta x)^{-(\theta+1)/\theta}\mathbf{1}_{(x\geq 0)}$ and Power (PW) distributions with density $\theta^{-1}x^{(1-\theta)/\theta}$, $x \in [0,1]$. These distributions are commonly considered in power studies of tests for exponentiality, in addition they give us a variety of situations which differ from the point of view of the hazard rate (and hence of MRL). We have DFR for Weibull with $\theta < 1$ and for J shaped distributions; IFR for Weibull with $\theta > 1$ and for LIFR distributions. The LN distributions have a hump shaped failure rate while a bathub shaped failure rate is present in PW distributions.

To judge how the performance of our test statistic depends on α, we chose the values $\alpha = 0.4$, 0.6, 0.75, 0.9, discarding $[n^\alpha]$ 'last' residual means.

As a yardstick for T_n^α, we report the power values of the classical Kolmogorov Smirnov statistic (KS) with estimated mean and the statistic L_n of Baringhaus and Henze (2000) which is a Kolmogorov-Smirnov type statistic based on an equivalent characterization of mean residual life and has been indicated as a good test for exponentiality in several situations. Similarly, for V_n^α we report the power values of the classical Cramer-von Mises statistic with estimated mean and the statistic G_n which depends on the quadratic distance and has also been proposed by Baringhaus and Henze (2000).

Table I and Table II show the estimates of power obtained for tests of size 0.05; in the last column of the tables, denoted HP($[n^\alpha]$), we report the highest power obtained by our test statistic for that distribution and in parenthesis the value of $[n^\alpha]$ by which this was obtained.

Table I. Estimated Power for T_n^α. Tests of size 0.05.

Alternative	n	$T_n^{0.4}$	$T_n^{0.6}$	$T_n^{0.75}$	$T_n^{0.9}$	KS	L_n	HP($[n^\alpha]$)
W(0.6)	10	44	40	17	2	36	25	44 (4)
	30	74	80	83	77	81	78	83 (13)
	50	85	92	95	96	96	95	96 (30)
W(1.6)	10	10	18	34	37	26	32	37 (7)
	30	04	35	66	77	71	78	77 (19)
	50	13	48	83	93	92	95	93 (30)
LN(0.8)	10	04	07	19	27	17	18	33 (8)
	30	05	05	16	45	46	40	45 (20)
	50	07	06	13	55	72	64	86 (39)
LN(1.2)	10	17	11	02	03	15	11	25 (1)
	30	51	46	28	07	35	34	51 (4)
	50	67	67	55	24	52	53	67 (5)
LIFR(0.5)	10	04	06	09	10	07	10	10 (7)
	30	01	07	13	16	13	16	16 (19)
	50	02	09	18	22	20	25	22 (30)
LIFR(3)	10	07	14	22	22	17	22	22 (7)
	30	03	25	44	48	43	54	48 (19)
	50	10	37	64	70	68	78	70 (30)
PW(1.4)	10	12	15	13	10	11	17	15 (5)
	30	24	46	34	20	29	46	46 (8)
	50	70	80	64	32	49	73	80 (10)
PW(2.0)	10	12	14	10	02	10	07	14 (4)
	30	07	15	19	26	22	17	26 (19)
	50	26	31	20	34	37	30	44 (39)
JS(0.5)	10	26	19	04	01	21	16	31 (1)
	30	63	60	47	37	52	52	63 (4)
	50	79	80	75	61	70	73	80 (8)
JS(1.0)	10	56	49	17	01	49	43	58 (2)
	30	92	93	90	85	90	90	93 (8)
	50	99	99	99	97	98	98	99 (12)

From the results in Table I we note that the choice of α has a pronounced effect on the power of the test T_n^α. This effect maybe different for different sample sizes and different alternatives. In the extensive simulations we ran, the trimming has always the effect of increasing the power. Generally this increases gradually up to a certain point and then decreases constantly afterwards.

Note that T_n^α can have considerably higher power than KS and L_n for small samples. The right choice of α always allows to at least match the power of the other competitors.

Table II. Estimated Power for V_n^α. Tests of size 0.05.

Alternative	n	$T_n^{0.4}$	$T_n^{0.6}$	$T_n^{0.75}$	$T_n^{0.9}$	CM	G_n	HP($[n^\alpha]$)
W(0.6)	10	45	38	30	02	40	43	45 (2)
	30	83	85	86	80	86	86	86 (13)
	50	94	96	97	97	98	98	97
W(1.6)	10	19	32	37	38	31	32	39 (6)
	30	62	73	80	83	81	83	83 (20)
	50	85	93	96	96	97	98	96
LN(0.8)	10	08	15	20	30	18	17	33(8)
	30	13	20	32	60	50	38	87(26)
	50	15	23	43	79	76	61	98(43)
LN(1.2)	10	19	08	04	03	16	18	25(1)
	30	51	40	27	06	38	43	53(1)
	50	68	63	50	15	57	61	70(2)
LIFR(0.5)	10	05	09	10	10	07	07	11(6)
	30	10	14	16	16	15	16	17(13)
	50	15	22	24	22	22	25	24(19)
LIFR(3)	10	12	20	22	22	19	20	23(7)
	30	43	50	54	51	51	55	55(13)
	50	64	75	78	72	76	81	78(19)
PW(1.4)	10	13	15	15	09	14	13	15(5)
	30	56	45	35	16	36	38	56(5)
	50	87	79	59	32	62	65	89(4)
PW(2.0)	10	10	13	13	02	11	9	13(5)
	30	16	15	18	27	27	15	29(22)
	50	35	27	22	39	48	28	50(39)
JS(0.5)	10	28	16	13	02	23	27	32(2)
	30	66	59	52	28	55	60	66(5)
	50	82	81	75	51	76	80	82(6)
JS(1.0)	10	57	44	33	01	52	55	60(2)
	30	94	93	92	78	92	94	94(5)
	50	99	99	99	96	99	99	99

The same holds slightly to a less extent for moderate or large sample sizes. Also, we note that the choice of α has generally less effect with increasing sample size obtaining high power for large values of α in most situations.

From Table II we note that the behavior of the test statistics V_n^α is similar to that of T_n^α. Again we have that the power of our test can be much higher than the CM and G_n statistics. Also, the quadratic statistic V_n^α has generally more power that the statistic T_n^α.

On the basis of these extensive simulations one can try to propose an empirical rule for the choice of α, the number of last residual means to discard.

The clearer situation seems to pertain to the case of IFR (and hence DMRL) distributions. In such a contest, either for T_n^α and V_n^α, a large value of α gets the highest power values and this happens for sample values of small, moderate and large size.

To a certain extent, the opposite happens for DFR distributions, i.e. small values of α obtain the largest power values for both T_n^α and V_n^α however, a wrong choice of α seems not to have a dramatic effect in moderate or large sample sizes.

The situation is more uncertain in the other two cases. For the Log Normal distribution increasing the value of the parameter θ has the effect of moving the hump of the failure rate towards the origin. This might explain why, for LN(1.2) small values of α are more efficient while for LN(0.8) the contrary happens.

For bathtub shaped FR the situation is more uncertain, in general however, middle values of α get the highest power.

The simulated values show that the test statistics T_n^α and V_n^α may perform very well. A drawback is however given by the necessity of choosing the trimming parameter α which maybe quite important for small or moderate sample sizes. The empirical results prompt us to study theoretically the efficiency of these test statistics by varying the parameter α for classes of alternatives. This problem revealed to be quite broad and pretty complex in itself and is under investigation.

4 Examples and discussion

As a first example on real data we consider a classical data set on the times, in operating hours, between successive failures of air conditioning equipment in an aircraft. These data are provided in Table III.

Table III: Operating hours between successive failures

90	10	60	186	61	49	14	24	56	20	79	84	44	59	29
118	25	156	310	76	26	44	23	62	130	208	70	101	208	

Lawless (1982) performed various tests of exponentiality on this data set, including Kolmogorov Smirnov, Cramer von Mises, Anderson Darling and specialized tests against IFR alternatives concluding that there was no evidence against the hypothesis of exponentiality. Successively, this data set has been used by Keating et Al. (1990) and Del Castillo and Puig (1999a) who consider tests of exponentiality against truly specific IFR models. They reject the null hypothesis with p values of .0384 and .027 respectively. We apply our test statistic to this data set. Given the empirical results of the previous section and the nature of the data, we apply the statistic V_n^α with a large number of residual means discarded. If we choose $[n^\alpha] = 23$, that is $\alpha \simeq 0.93$ the statistic V_n^α compares the first six sample residual means obtaining the value $V_n^\alpha = 0.0767$, an approximate p value for this result, calculated on the basis of 10000 samples is 0.035. The statistics G_n of Baringhaus and Henze obtains a p value around 0.25.

The second data set we consider is from Kotz and Johnson (1983) and represents the survival times (in days) after diagnosis of 43 patients with a certain kind of leukemia. For such a data set, IFR maybe too restrictive. Hopefully the treatment, applied after diagnosis, will (at least for a period) decrease the failure rate.

Table IV: Survival times in days after diagnosis.

7	47	58	74	177	232	273	285	317	429	440	445
455	468	495	497	532	571	579	581	650	702	715	779
881	900	930	968	1077	1109	1314	1334	1367	1534	1712	1784
1877	1886	2045	2056	2260	2429	2509					

If the Hollander and Proschan (1972) test against NBU alternatives is applied, it obtains a p value of 0.07. Del Castillo and Puig (1999b) apply to this data set a likelihood ratio test against singly truncated normal alternatives which are recommended for lifetime data whose nature suggests an IFR distribution with hazard rate not vanishing at 0. Their statistics reject the hypothesis with a p value 0.033. Again, in this case we may try to apply the statistic V_n^α with a large number of last residual means discarded. Choosing $[n^\alpha] = 30$, i.e. $\alpha \simeq 0.9$ we obtain $V_n^\alpha = 0.1923$ which rejects the hypothesis of exponentiality with a p value of 0.0328. The Cramer von Mises test and the statistic G_n do not reject in this case with p values respectively of 0.149 and 0.107.

The new tests statistics T_n^α and V_n^α that we have proposed seem to work well even in situations where traditional tests of exponentiality fail to detect departures from the null hypothesis, in particular, for IFR or even more general classes of distributions where its performance seems comparable to that of specialized tests. A drawback of our statistics is given by the necessity of the choice of the number of residual means that we need to compare. Simulations have shown that this has definitely an effect on the power of the test statistics. If, in practice one has some idea of the possible model underlying the data one can use the empirical rules given above; alternatively, in order to try to distribute power more evenly, one could try to use combinations of two or more test statistics $T_n^{\alpha_i}$, $i = 1, \dots k$ and reject the null hypothesis when at least one of them rejects the hypothesis.

5 Proofs

Proof of Theorem 1. For $t \in [X_{(k)}, X_{(k+1)})$ we write

$$\bar{X}_{>t} = \frac{1}{n-k+1} \sum_{k+1}^{n+1} (X_{(i)} - t) \tag{5.19}$$

$$= \frac{1}{n-k+1} [(n+1)\bar{X} - D_{n+1}(t)] \tag{5.20}$$

$$= \left[\bar{F}_{n+1}(t)\right]^{-1} \left[\bar{X} - \frac{D_{n+1}(t)}{n+1}\right]. \tag{5.21}$$

where $D_{n+1}(t)$ has been defined in (2.13). Note that from Koul (1978) we have that $D_{n+1}(t)/(n+1) \stackrel{a.s.}{\to} \int_0^t \bar{F}(x)dx$ and by the Glivenko-Cantelli theorem $\bar{F}_{n+1}(t) \stackrel{a.s.}{\to} \bar{F}(t)$ uniformly in t. Then we have

$$\bar{X} - \bar{X}_{>t} \stackrel{a.s.}{=} \theta - m(t), \quad \forall t.$$

The results of the theorem then follows from standard devices if the difference $\bar{X}_{>t} - \bar{X}_{>k}$ can be made arbitrarily small in probability for all $t > 0$ in the interval $[X_{(k)}, X_{(k+1)})$. To this end note that, for $t \in [X_{(k)}, X_{(k+1)})$,

$$\bar{X}_{>t} - \bar{X}_{>k} = t - X_{(k)}$$

From continuity of F this difference can be made as small as desired in probability, in fact we have

$$|F(t) - F(X_{(k)})| \leq |F(X_{(k+1)}) - F(X_{(k)})| \leq \left| U_{(k+1)} - \frac{k+1}{n+1} \right| + \left| U_{(k)} - \frac{k}{n+1} \right| + \frac{1}{n+1}$$

that the right hand side of the above inequality converges in probability to 0 follows from the law of large numbers and the representation of $U_{(k)}$ as $S(k)/S(n+1)$, where $S(k)$ has been defined above as the sum of k *i.i.d.* exponential r.v.'s. From this we obtain

$$\bar{X}_{>k} \stackrel{p}{\to} \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}, \quad t \in [X_{(k)} - X_{(k+1)})$$

The last step is to note that the inequality

$$\frac{\int_{X_{(n-[n^\alpha])}}^\infty \bar{F}(x)dx}{\bar{F}(t)} < \varepsilon$$

holds for arbitrary $\varepsilon > 0$ as $n \to \infty$.

The proof of Theorem 2 relies on the representation of T_n^α and V_n^α in terms of a uniform quantile process in a weighted metric. Note in fact that by using the same representation as in (2.14) and after some manipulation we have

$$\frac{\bar{X} - \bar{X}_{>k}}{\bar{X}} \quad \stackrel{D}{=} \quad \frac{\frac{i}{n+1} - U_{(i)}}{\frac{i}{n+1}}$$

for $k = 1, \ldots n$, $i = n - k + 1$ and where $U_{(i)}$ is the i-th order statistics from a uniform random sample. The relevant asymptotic theory can be obtained by exploiting results on uniform empirical processes in weighted metrics. To this end we define the functions

$$U_n(t) = \begin{cases} U_{(i)} & \frac{i}{n+2} < t \leq \frac{i+1}{n+2}, \quad i = 1 \ldots n, \\ 0 & \text{otherwise} \end{cases}$$

$$q_n(t) = \begin{cases} \frac{i}{n+1} & \frac{i}{n+2} < t \le \frac{i+1}{n+2}, \quad i = 1 \dots n, \\ 0 & \text{otherwise} \end{cases}$$

The function $[q_n(t) - U_n(t)]/q_n(t)$ is a step function with jump points in $i/(n+2)$, $i = 1, \dots n$, hence, if we define a process

$$\tilde{u}_n(t) = \sqrt{n}[q_n(t) - U_n(t)]$$

then it holds that

$$\sqrt{n}\, T_n^\alpha \overset{D}{=} \sup_{\frac{n^\alpha}{n+2} < t \le \frac{n+1}{n+2}} \left| \frac{\tilde{u}_n(t)}{q_n(t)} \right|$$

and

$$(n+2)\, V_n^\alpha \overset{D}{=} \int_{\frac{n^\alpha}{n+2}}^{\frac{n+1}{n+2}} \left| \frac{\tilde{u}_n(t)}{q_n(t)} \right|^2 dt.$$

Next we define a continuous version of the uniform quantile process as

$$u_n(t) = \sqrt{n}[t - U_n(t)]$$

It is clear from the definition that $u_n(\frac{i}{n+1}) = \tilde{u}_n(t)$, $\frac{i}{n+2} < t \le \frac{i+1}{n+2}$, $i = 1 \dots n$. The asymptotic distributions of T_n^α and V_n^α will then be obtained by those of the corresponding functionals of the weighted uniform quantile process $u_n(t)/t$. We will adapt the methods discussed in Csörgő and Horváth (1993) by providing suitable approximations by Gaussian processes. Since the weak convergence of $u_n(t)$ in weighted metric does not imply the the convergence in distribution of supremum or quadratic functionals for the weight function $q(t) = t$, we will study the distributional properties of T_n^α, V_n^α on their own. Before doing so we need some preliminary results that, for convenience, we recall here in the form of lemmas.

Lemma 1. *We can define a sequence of Brownian Bridges $\{B_n(t), 0 \le t \le 1\}$ such that*

$$\sup_{0 \le t \le 1} |\tilde{u}_n(t) - B_n(t)| \overset{a.s.}{=} O(n^{-1/2} \log n) \tag{5.22}$$

and

$$n^{1/2-v} \sup_{\lambda/n \le t \le 1} \frac{|\tilde{u}_n(t) - B_n(t)|}{t^v} = O_p(1) \tag{5.23}$$

for all $0 < v \le 1/2$ and $0 < \lambda < \infty$.

This lemma is just a straightforward modification of Theorem 4.4.2 in Csörgő and Horváth (1993) which can be obtained by noting that

$$\sup_{0<t<1} |u_n(t) - \tilde{u}_n(t)| \le \frac{\sqrt{n}}{n+2}.$$

Lemma 2. (Csörgő and Horváth, 1993, p.302 Theorem 2.3, *iv*). *Let* $a(n) \to 0$, $na(n) \to \infty$ *as* $n \to \infty$. *Then*

$$a(n)^{1/2} \sup_{a(n) \le t \le 1-a(n)} \frac{|u_n(t)|}{t} \xrightarrow{D} \sup_{0 \le t \le 1} |W(t)|. \tag{5.24}$$

Next, we need a few properties of some Gaussian processes. We say that $\{V(t),\ -\infty < t < \infty\}$ in an Ornstein-Uhlenbeck process if it is Gaussian with $EV(t) = 0$, and $EV(t)V(s) = \exp\{-|t-s|\}$. Also, we note that if $\{W(t),\ 0 < t < \infty\}$ is a Wiener process, computing the covariances of the processes we can show that

$$\left\{ \frac{W(t)}{t^{1/2}},\ 0 < t < \infty \right\} \stackrel{D}{=} \left\{ V(\frac{1}{2}\log t),\ 1 < t < \infty \right\} \tag{5.25}$$

We will need the following result

Lemma 3. We assume that $0 < p < \infty$. As $T \to \infty$, we have

$$\left\{ \int_0^T |V(t)|^p \, dt - mT \right\} / (DT)^{1/2} \xrightarrow{D} N(0,1), \tag{5.26}$$

where $N(0,1)$ is a standard normal r.v.,

$$m = m(p) = (2\pi)^{-1/2} \int_{\infty}^{\infty} |x|^p \exp(-x^2/2) dx$$

and

$$D = D(p) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} \int_{\infty}^{\infty} |xy|^p$$
$$\times \left\{ \frac{1}{2\pi(1-e^{-2|u|})^{1/2}} \exp\left(-\frac{x^2+y^2-2e^{-|u|}xy}{2(1-e^{-2|u|})} \right) - \phi(x)\phi(y) \right\} dx\, dy\, du$$

See, for example Csörgő and Horváth (1993), p.318. We have now all the necessary tools to provide a proof of Theorem 2.

Proof of Theorem 2.

i) we will first show that

$$\sup_{\frac{n^\alpha}{n+2} < t \le \frac{n+1}{n+2}} |n^{\alpha/2} T_n^\alpha(t) \ - \ n^{\alpha/2-1/2} \frac{u_n(t)}{t}| = o_p(1) \tag{5.27}$$

To this end note that by the definition of $\tilde{u}_n(t)$ and $u_n(t)$ we can rewrite the above expression as

$$
\begin{aligned}
n^{\alpha/2} \sup_{\frac{n^\alpha}{n+2}<t\leq\frac{n+1}{n+2}} \left|\frac{U_n(t)}{t} - \frac{U_n(t)}{q_n(t)}\right| &= n^{\alpha/2} \max_{n^\alpha<i\leq n} U_{(i)} \sup_{\frac{i}{n+2}<t\leq\frac{i+1}{n+2}} \left|\frac{1}{t} - \frac{n+1}{i}\right| \\
&= n^{\alpha/2} \max_{n^\alpha<i\leq n} U_{(i)} \left[\max\left(\frac{n-i+1}{i(i+1)}, \frac{1}{i}\right)\right] \\
&= n^{\alpha/2} \max\left[\max_{n^\alpha<i\leq n/2} U_{(i)} \frac{n-i+1}{i(i+1)}, \max_{n/2<i\leq n} U_{(i)}\frac{1}{i}\right]
\end{aligned}
$$

and since $U_{(i)} \xrightarrow{p} i/(n+1)$, the above expression can be seen to be $n^{\alpha/2}O_p(n^{-\alpha}) = o_p(1)$. Next, from the fact that $\sup_{0\leq t\leq 1}|u_n(t)|$ is bounded in probability we easily see that

$$
n^{\alpha/2-1/2} \sup_{1-\frac{n^\alpha}{n}\leq t\leq 1} \frac{|u_n(t)|}{t} = o_p(1),
$$

hence result (2.17) follows from applying Lemma 2 with $a(n) = n^\alpha/n$.

ii) let $a(n,\alpha) = n^\alpha/(n+2)$ we will first show that

$$
\int_{a(n,\alpha)}^1 \left|\left|\frac{\tilde{u}_n(t)}{q_n(t)}\right|^2 - \left|\frac{B_n(t)}{t}\right|^2\right| dt = o_p(1). \tag{5.28}
$$

The above term is not greater than

$$
\int_{a(n,\alpha)}^1 \left|\left|\frac{\tilde{u}_n(t)}{q_n(t)}\right|^2 - \left|\frac{\tilde{u}_n(t)}{t}\right|^2\right| dt + \int_{a(n,\alpha)}^1 \left|\frac{\tilde{u}_n(t)^2 - B_n(t)^2}{t^2}\right| dt = A_{1,n} + A_{2,n}
$$

We can make the following estimates:

$$
\begin{aligned}
A_{1,n} &= \int_{a(n,\alpha)}^1 \frac{\tilde{u}_n(t)^2}{q_n(t)^2}\left|1 - \frac{q_n(t)^2}{t^2}\right| dt \\
&\leq \left[a(n,\alpha)^{1/2} \sup_{a(n,\alpha)<t<1} \frac{|\tilde{u}_n(t)|}{q_n(t)}\right]^2 a(n,\alpha)^{-1} \int_{a(n,\alpha)}^1 \left|1 - \frac{q_n(t)^2}{t^2}\right| dt \\
&= O_p(1)\, O\left(\frac{(1-\alpha)\log n}{n^\alpha}\right)
\end{aligned}
$$

where we have used result (2.17) and the fact that

$$
\int_{a(n,\alpha)}^1 \left|1 - \frac{q_n(t)^2}{t^2}\right| dt = O\left(\frac{(1-\alpha)\log n}{n}\right).
$$

Before considering the term $A_{2,n}$ we note that

$$|g^2 - h^2| \leq 4|g-h|^2 + 4|h||g-h| \tag{5.29}$$

by using the above inequality we estimate

$$\begin{aligned} A_{2,n} &\leq 4\int_{a(n,\alpha)}^{1} \frac{|\tilde{u}_n(t) - B_n(t)|^2}{t^2}dt + 4\int_{a(n,\alpha)}^{1} \frac{|B_n(t)||\tilde{u}_n(t) - B_n(t)|}{t^2}dt \\ &= A_{21,n} + A_{22,n} \end{aligned}$$

Now, for $v > 0$,

$$\begin{aligned} A_{21,n} &\leq 4\left[\sup_{a(n,\alpha)<t<1} \frac{|\tilde{u}_n(t) - B_n(t)|}{t^{1/2-v}}\right]^2 \int_{a(n,\alpha)}^{1} t^{-(1+2v)}dt \\ &= O_p(1)n^{-2v}\int_{a(n,\alpha)}^{1} t^{-(1+2v)}dt \\ &= O_p(1)\, O(n^{-2\alpha v}) \end{aligned}$$

by applying (5.23) of Lemma 1 with $\lambda = n^{\alpha}$. Again, Lemma 1 leads us to the following estimate for $A_{22,n}$,

$$\begin{aligned} A_{22,n} &\leq 4\sup_{a(n,\alpha)<t<1} \frac{|\tilde{u}_n(t) - B_n(t)|}{t^{1/2-v}} \int_{a(n,\alpha)}^{1} \frac{|B_n(t)|}{t^{(3/2+v)}}dt \\ &= O_p(1)n^{-v}\int_{a(n,\alpha)}^{1} \frac{|B_n(t)|}{t^{(3/2+v)}}dt. \end{aligned}$$

In order to evaluate the above term note that, by the definition of a Brownian Bridge, we have

$$\begin{aligned} E\int_{a(n,\alpha)}^{1} \frac{|B_n(t)|}{t^{(3/2+v)}}dt &= E\int_{a(n,\alpha)}^{1} \frac{|W_n(t) - tW_n(1)|}{t^{(3/2+v)}}dt \\ &\leq E\int_{a(n,\alpha)}^{1} \frac{|W_n(t)|}{t^{(3/2+v)}}dt + E\int_{a(n,\alpha)}^{1} \frac{|W_n(1)|}{t^{(1/2+v)}}dt \\ &= E|W_n(1)|\int_{a(n,\alpha)}^{1} t^{-(1+v)}dt + E|W_n(1)|\int_{a(n,\alpha)}^{1} t^{-(1/2+v)}dt \\ &= E|N(0,1)|\, O(a(n,\alpha)^{-v}) \end{aligned}$$

where we have used the self similarity property of the Wiener process and the fact that $W_n(1)$ is a standard normal r.v. for any n. From this it follows that

$$a(n,\alpha)^v \int_{a(n,\alpha)}^{1} \frac{|B_n(t)|}{t^{(3/2+v)}}dt = O_p(1)$$

and hence

$$n^{-v} \int_{a(n,\alpha)}^{1} \frac{|B_n(t)|}{t^{(3/2+v)}} dt = o_p(1).$$

Hence we have proved (5.28). The result will follow if we show that $\int_{a(n,\alpha)}^{1} B_n(t)^2/t^2\, dt$ properly normalized, converges to a standard normal r.v.. Since $\{B(t), 0 \leq t \leq 1\} \stackrel{D}{=} \{W(t) - tW(1), 0 \leq t \leq 1\}$ it suffices to prove our statement when $B(t)$ is replaced by $W(t) - tW(1)$. Applying (5.29) again we have

$$\begin{aligned}
&[(1-\alpha)\log n]^{-1/2} \int_{a(n,\alpha)}^{1} \frac{|W(t)^2 - B(t)^2|}{t^2} dt \\
&= [(1-\alpha)\log n]^{-1/2} \int_{a(n,\alpha)}^{1} \frac{|W(t)^2 - |W(t) - tW(1)|^2|}{t^2} dt \\
&\leq [(1-\alpha)\log n]^{-1/2} \left[4W(1)^2 \int_{a(n,\alpha)}^{1} dt + 4W(1) \int_{a(n,\alpha)}^{1} \frac{W(t)}{t} dt \right] \\
&= [(1-\alpha)\log n]^{-1/2} 4W(1)^2 \left[\int_{a(n,\alpha)}^{1} dt + \int_{a(n,\alpha)}^{1} t^{-1/2} dt \right] \\
&= o_p(1).
\end{aligned}$$

Next, using (5.25) we note that

$$\int_{a(n,\alpha)}^{n+1/n+2} \frac{W(t)^2}{t^2} dt \stackrel{D}{=} \int_{a(n,\alpha)}^{n+1/n+2} \frac{1}{t} V\left(\frac{1}{2}\log t\right) dt \tag{5.30}$$

by applying the transformation $u = 1/2 \log t$ and using stationarity of the Ornstein-Uhlenbeck process we have that (5.30) is distributionally equivalent to

$$2 \int_{0}^{1/2 \log \frac{n+1}{n^\alpha}} V(u)^2 du$$

result (2.18) now follows from an application of Lemma 3 with the normalizing constants $D = 2$ and $m = 1$.

6 References

Alwasel,I. (2001). On goodness of fit testing for exponentiality using the memoryless property. *J. of Nonparametric Statistics* **13**, 569-581.

Bandyopadhyay, D. and Basu, A.P. (1990). A class of tests for exponentiality against decreasing mean residual life alternatives. *Comm. in Stat. - Theory and Methods* **19**, 905-920.

Baringhaus, L. and Henze, N. (1991). "A class of consistent tests for exponentiality based on the empirical Laplace transform," *Ann. Inst. Statist. Math.* **43**, 551-564.

Baringhaus,L. and Henze,N. (1992). "An Adaptive Omnibus Test for Exponentiality," *Communications in Statistics - Theory and Methods* **21**, 969-978.

Baringhaus, L. and Henze, N. (2000). "Tests of fit for exponentiality based on a characterization via the mean residual life function," *Statistical Papers* **41**, 225-236.

Bergman, B. and Klefsjö, B. (1989). A family of test statistics for detecting monotone mean residual life. *J. Stat. Plann. and Infer.* 21, 161-178.

Bhattacharjee, M.C. and Sen, P.K. (1995). Kolmogorov -Smirnov type tests for NB(W)UE alternatives under censoring schemes. In Koul, H.L. and Deshpande, J.V. (Eds.) *Analysis of Censored Data*, 25-38.

Csörgő,M. and Horváth, L. (1993). *Weighted approximations in probability and statistics.* Wiley, New York.

Del Castillo, J. and Puig, P. (1999a). Invariant exponential models applied to reliability theory and survival analysis. *J. Amer. Statist. Assoc.*, **94**, 522-528.

Del Castillo, J. and Puig, P. (1999b). The best test of exponentiality against singly truncated normal alternatives. *J. Amer. Statist. Assoc.*, **94**, 529-532.

Doksum, K.A. e Yandell, B.S. (1984). "Tests for exponentiality," *Handbook of statistics*, Vol. 4, North Holland, 579-612.

Ebrahimi, N., Soofi, E.S. and Habibullah, M. (1992). "Testing Exponentiality Based on Kullback-Leibler Information," *Journal of the Royal Statistical Society, Ser. B* **54**, 739-748.

Gatto, R. and Jammalamadaka, S. Rao (2002). "A saddlepoint approximation for testing exponentiality against some increasing failure rate alternatives," *Statistics and Probab. Letters* **58**, 71-82.

Grzegorzewski, P. and Wieczorkowski, R. (1999). "Entropy-Based Goodness of Fit Test for Exponentiality," *Communications in Statistics - Theory and Methods* **28**, 1183-1202.

Grubbs, F.E. (1971) Fiducial bounds on reliability for the two parameter negative exponential distribution. *Technometrics* **13**, 873-876.

Guess, F. and Proschan, F. (1988). Mean residual life, theory and applications. In Krishnaiah, P.R. and Rao, C.R. (Eds.), Handbook of Statistics Vol. 7, *Reliability and Quality Control*, 215-224.

Gupta, R.D. and Richards, D.St.P. (1997). Invariance properties of some classical tests for exponentiality. *J. Stat. Plann. Inference* **63**, 203-213.

Henze, N. and Meintanis, S.G. (2002). Tests of fit for Exponentiality based on the empirical Laplace transform. *Statistics* **36**, 147-162.

Henze, N. (1993). "A New Flexible Class of Omnibus Tests for Exponentiality," *Communications in Statistics - Theory and Methods* **22**, 115-133.

Hollander, M. and Proschan, F. (1972). Testing whether New is Better than Used. *Ann. Math. Stat.* **43**, 1136-1146.

Jammalamadaka,S.Rao and Lee,E.S. (1998). "Testing for Harmonic New Better than Used in Expectation", *Probab. in Engg. and Inform. Sci.* **12**, 409-416.

Jammalamadaka,S.Rao and Taufer, E. (2001). Testing Exponentiality by comparing the Empirical Distribution Function of the Normalized Spacings with that of the Original Data. Preprint.

Keating, J.P., Glaser, R.E. and Ketchum, N.S. (1990). Testing hypotheses about the shape of a Gamma distribution. *Technometrics*, **32**, 67-82.

Klar, B. (2000). "A class of tests for exponentiality against HNBUE alternatives," *Statistics and Prob. Letters* **47**, 199-207.

Kotz. S. and Johnson, N.L. (1983). *Encyclopedia of Statistical Sciences*, Vol. 3. Wiley, New York.

Koul, H.L. (1978). Testing for new is better than used in expectation. Comm. Statist. - Theor. Meth. A **7**, 685-701.

Lawless, J.F. (1982). *Statistical Models and Methods for Lifetime Data.* Wiley, New York.

Patel, J.K. (1983). Hazard rate and other classifications of distributions. In Johnson, N.L. and Kotz, S, Eds. *Encyclopedia od Statistical Sciences.* Wiley, New York.

Shanbhag, D.N. (1970). The Characterizations for Exponential and Geometric Distributions. *J. Amer. Statist. Assoc.* , 65, 1256-1259.

Shapiro, S.S. (1995). Goodness of Fit tests. In *The Exponential Distribution: Theory Methods and Applications.* Balakrishnan, N. and Basu, A.P. Editors, Gordon and Breach, Amsterdam.

Taufer, E. (2000). A new test for exponentiality against omnibus alternatives. *Stochastic Modelling and Applications* **3**, 23-36.